Bernstein-Sato polynomials in positive characteristic

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• "Bernstein-Sato theory for arbitrary ideals in positive characteristic" arXiv1907.07297
• "Bernsetin-Sato roots for monomial ideals in prime characteristic" arXiv1907.11709

Story over $\mathbb{C}$

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Theorem (Kashiwara, Budur-Mustață-Saito)

The roots of $b_{\mathfrak{a}}(s)$ are rational and negative.

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We will only be able to define an analogue of the roots.

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$\mathrm{minpoly}(s_1 \curvearrowright N) \in k[s]$, and we want to encode information about F-jumping numbers of $\mathfrak{a}$ $(\subseteq \mathbb{Q})$

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now we get an infinite family

$$s_{p^i} := -\sum_{\substack{(a_1,\dots,a_r)\in\mathbb{N}_0^r \\ a_1+\cdots+a_r=p^i}} \partial_{t_1}^{[a_1]} t_1^{a_1} \cdots \partial_{t_r}^{[a_r]} t_r^{a_r} \quad (i = 0, 1, 2, \dots)$$

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(1) $s_{p^i} \in (D^{i+1}_{R[t_1,\ldots,t_r]})_0$ and thus $s_{p^0}, s_{p^1}, \ldots, s_{p^{e-1}} \curvearrowright N^e$.

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(2) For all i, j, $s_{p^i} s_{p^j} = s_{p^j} s_{p^i}$ for all i, j.
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$$\implies N^e = \bigoplus_{\alpha \in \mathbb{F}_p^e} N^e_\alpha$$

where, for $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_{e-1}) \in \mathbb{F}_p^e$,
$N^e_\alpha = \{u \in N^e : s_{p^i} \cdot u = \alpha_i u \text{ for all } i = 0, 1, \ldots, e-1\}$.

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Summary:

$$\begin{array}{ccccccc} N^1 & \longrightarrow & N^2 & \longrightarrow & N^3 & \longrightarrow & \cdots \longrightarrow \lim_{\to e} N^e =: N \\ \| & & \| & & \| & & \\ \bigoplus_{\alpha \in \mathbb{F}_p} N^1_\alpha & \longrightarrow & \bigoplus_{\alpha \in \mathbb{F}_p^2} N^2_\alpha & \longrightarrow & \bigoplus_{\alpha \in \mathbb{F}_p^3} N^3_\alpha & \longrightarrow & \cdots \end{array}$$

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Theorem (QG19)

We have

$$N = \bigoplus_{\alpha \in \mathbb{F}_p^{\mathbb{N}}} N_\alpha$$

where, for $\alpha = (\alpha_0, \alpha_1, \dots)$, $N_\alpha := \{u \in N : s_{p^i} \cdot u = \alpha_i u \text{ for all } i \in \mathbb{N}_0\}$.
Moreover, $\#\{\alpha \in \mathbb{F}_p^{\mathbb{N}} : N_\alpha \neq 0\} < \infty$.

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When char $k = p > 0$

$$N \circlearrowleft s_{p^0}, s_{p^1}, s_{p^2}, \ldots$$

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Definition

A p-adic integer $\alpha = \alpha_0 + p\alpha_1 + p^2\alpha_2 + \cdots \in \mathbb{Z}_p$ (where $\alpha_i \in \{0, 1, \ldots, p-1\}$) is a **Bernstein-Sato root** of $\mathfrak{a}$ if $N_{(\alpha_0, \alpha_1, \ldots)} \neq 0$.

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(3) Let $\mathfrak{a} \subseteq \mathbb{Z}[x_1, \ldots, x_n]$ be a monomial ideal. Then for all $p \gg 0$ we have

$$BS(\mathfrak{a}_p) = \{\text{Roots of } b_{\mathfrak{a}_{\mathbb{C}}}(s)\}.$$

Results

Connection to ν-invariants

•Given an ideal $J \subseteq R$ with $\mathfrak{a} \subseteq \sqrt{J}$, and an integer $e > 0$, Mustaţă, Takagi and Watanabe defined

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Proposition (QG19)

We have

$$BS(\mathfrak{a}) = \bigcap_{e=0}^{\infty} \overline{\nu_{\mathfrak{a}}^{\bullet}(p^e)}$$

where $(\bar{\ })$ *denotes p-adic closure.*

Examples

Example 1: Suppose $R = k[x, y]$ and $\mathfrak{a} = (x^2 + y^3)$.

(a) When $k = \mathbb{C}$, $b_{\mathfrak{a}}(s) = (s + \frac{5}{6})(s + 1)(s + \frac{7}{6})$.

(b) When char $k \equiv 1 \mod 3$ then $BS(\mathfrak{a}) = \{-5/6, -1\}$

(c) When $2 \neq$ char $k \equiv 2 \mod 3$ then $BS(\mathfrak{a}) = \{-1\}$.

Examples

Example 2: Suppose $R = k[x, y]$ and $\mathfrak{a} = (x^2, y^3)$.

(a) When $k = \mathbb{C}$,
$b_{\mathfrak{a}}(s) = (s + 5/6)(s + 7/6)(s + 4/3)(s + 3/2)(s + 5/3)(s + 2)$.

(b) When char $k = 2$, $BS(\mathfrak{a}) = \{-4/3, -5/3, -2\}$.

(c) When char $k = 3$, $BS(\mathfrak{a}) = \{-3/2, -2\}$.

(d) When char $k > 3$, $BS(\mathfrak{a}) = \{-5/6, -7/6, -4/3, -3/2, -5/3, -2\}$.

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(2) Can we define an analogue of multiplicity in characteristic $p > 0$?

Thank you for your attention